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EULER AND NAVIER-STOKES EQUATIONS

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ABSTRACT

Implicit compact finite difference schemes for the Euler equations are described which furnish equivalent treatments of the conservation and nonconservation forms; a simple modification yields an entropy-producing scheme. An extension of the scheme also treats the compressible Navier-Stokes equations; when the viscosity and heat conduction coefficients are negligible only the boundary data appropriate to the Euler equation influence the solution to any significant extent, a result consistent with singular perturbation theory.

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This paper discusses a class of compact finite difference schemes for the Euler and Navier-Stokes equations. These schemes are closely related to schemes described in Philips and Rose [4] but are developed here with specific reference to hydrodynamics in order to treat important details which are not immediately evident in a more general mathematical setting. However, solution methods which are described in [4] also apply to the schemes considered here and, where appropriate, we refer the reader to that paper for further details.

The paper is divided into two essentially separate but, nevertheless, closely related parts. Part I treats the Euler equations and shows the formal equivalence betweeen solutions of the conservation and nonconservation forms of a compact finite difference scheme which arises from a consistent use of the product rule for differences. A slight modification yields an entropy-producing scheme. Part II extends the scheme to the Navier-Stokes equations, our principle objective being to demonstrate that, as the coefficients of viscosity and heat conduction vanish, only the boundary conditions for the associated Euler problem influence the solution, a result consistent, therefore, with singular perturbation arguments.

Part I

I.1. Introduction

Smooth solutions of systems of hyperbolic equations in the conservation form

$$U_t + F_x(U) = 0, \tag{1.1}$$

also satisfy

$$U_t + A(U)U_x = 0, \tag{1.2}$$

where $A = \text{grad } F$. Equation (1.1) also permits weak solutions, i.e., solutions which are smooth except along certain curves $x = x(t)$ across which

$$[U]\dot{x} - [F] = 0, \tag{1.3}$$

where [ ] indicates the jump in value across $x = x(t)$. Generally, solutions satisfying (1.3) are not uniquely determined; for problems of physical interest a unique solution can be determined by the requirement that an entropy condition also be satisfied across the discontinuity. Such a solution may also be characterized as the weak limit of smooth solutions of

$$U_t + A(U)U_x = \nu U_{xx}, \tag{1.4}$$

as $\nu \to 0$.

In applications to inviscid fluid dynamics (1.1) are the Euler equations which express the conservation of mass, momentum, and energy; if $x = x(t)$ is a shock discontinuity, (1.3) express the Rankine-Hugoniot conditions. Equation (1.4) is a model of the Navier-Stokes equations for which (1.1) is the inviscid singular perturbation limit ($\nu \to 0$).

These observations play an important role in computational fluid dynamics. The artificial viscosity method of von Neumann and Richtmyer [5] is based upon a variant of (1.4) and has been a widely employed shock-capturing finite difference technique. The work of Lax [2] and others has emphasized the fact that certain finite difference schemes having the conservation property expressed by (1.1) also converge to the physically relevant discon-

tinuous solution of (1.1). For one dimensional problems such conservation-preserving difference schemes have proved to be more effective than artificial viscosity methods in reducing the spread of the numerical solution at discontinuities. In higher dimensions no completely satisfactory methods for shock capturing are yet available.

In a recent study of a class of implicit compact finite difference schemes (Philips and Rose [4]) a Riemann problem was treated with reasonable accuracy in spite of the fact that the scheme was, formally, nonconservative. Part I of this paper clarifies this result. We are able to show by purely formal arguments that when a leapfrog scheme is used to express the conservation form of the Euler equations certain natural auxiliary conditions transform the system to an equivalent nonconservative form. We are also able to describe a modification of the scheme which is entropy-producing (i.e., physically dissipative).

2. The Euler Equations

The Euler equations in one dimension may be described as follows: define

$$\dot{D}\phi \equiv \partial_t \phi + \partial_x (u\phi), \qquad D_t \phi \equiv \partial_t \phi + u\partial_x \phi. \tag{2.1}$$

Equations expressing conservation of mass, momentum, and energy in a domain Ω in the (x,t) plane are given by

$$\dot{D}\rho = 0, \qquad \dot{D}\rho u + \partial_x p = 0, \qquad \dot{D}\rho E + \partial_x (pu) = 0 \tag{2.2}$$

in which ρ = density, u = velocity, E = total specific energy,

p = pressure. If e = specific internal energy then $E = u^2/2 + e$ and $p = (\gamma-1)\rho e$, the latter expressing the equation of state of a perfect gas with gas constant γ.

Consider smooth solutions of (2.2); since $\dot{D}\rho = 0$ then $\dot{D}\rho\phi = \rho D_t\phi$ and (2.2) may be transformed to the nonconservation form

$$\dot{D}\rho = 0$$

$$\rho D_t u + \partial_x p = 0 \tag{2.3}$$

$$\rho D_t E + p\partial_x u + u\partial_x p = 0$$

If use is made of the momentum equation and the relationship $\frac{1}{2} D_t u^2 = uD_t u$ the energy equation in (2.3) can be expressed in the form

$$\rho D_t e + p\partial_x u = 0.$$

Next, employ the first law of thermodynamics in the form

$$de = Tds - pd\rho^{-1} \tag{2.4}$$

where s = entropy and T = temperature to obtain

$$\rho D_t e + p\partial_x u = \rho T D_t s. \tag{2.5}$$

Thus, for smooth nonviscous flows the energy equation is equivalent to

$$D_t s = 0. \tag{2.6}$$

3. A Compact Finite Difference Scheme

The equivalence of (2.2) and (2.3) depends upon the product rule for differentiation

$$\partial(\phi\psi) = \phi\partial\psi + \psi\partial\phi$$

for smooth functions. A similar product rule holds for difference operators and it is reasonable to examine the formal consequences of employing the rule in finite difference equations.

In order to do so let δ_x, δ_t denote central divided difference operators and μ_x, μ_t central averaging operators on a mesh whose characteristic length is h. The product rule for differences is

$$\delta(\phi\psi) = (\mu\phi)\delta\psi + (\mu\psi)\delta\phi.$$

Define

$$\begin{aligned} \overset{\bullet}{D}{}^h\phi &\equiv \delta_t\phi + \delta_x(u\phi), \\ D_t^h\phi &\equiv \delta_t\phi + (\mu_x u)\delta_x\phi. \end{aligned} \tag{3.1}$$

Then

$$\overset{\bullet}{D}{}^h(\phi\psi) = (\mu_t\phi)\overset{\bullet}{D}{}^h\psi + (\mu_t\psi)D_t^h\phi + \left(\mu_x\phi-\mu_t\phi\right)\delta_x(\phi u) + \left(\mu_x(\phi u)-(\mu_t\phi)(\mu_x u)\right)\delta_x\psi, \tag{3.2}$$

and

$$\tfrac{1}{2} D_t^h\phi^2 = (\mu_t\phi)D_t^h\phi + (\mu_x u)(\mu_x\phi-\mu_t\phi)\delta_x\phi. \tag{3.3}$$

Immediate consequences are:

Lemma 1: If

$$(\mu_t\rho)(\mu_x u) = \mu_x(\rho u), \qquad \overset{\bullet}{D}{}^h\rho = 0,$$

then

$$\overset{\bullet}{D}{}^h(\rho\phi) = (\mu_t\rho)D_t^h\phi + (\mu_x\phi-\mu_t\phi)\delta_x(\rho u).$$

Lemma 2: If

$$\mu_t u = \mu_x u,$$

then

$$\tfrac{1}{2} D_t^h u^2 = (\mu_t u)D_t^h u.$$

Next, partition the fundamental domain Ω by cells $\{\pi_i^n\}$ to obtain Ω_h; a a typical cell is shown below:

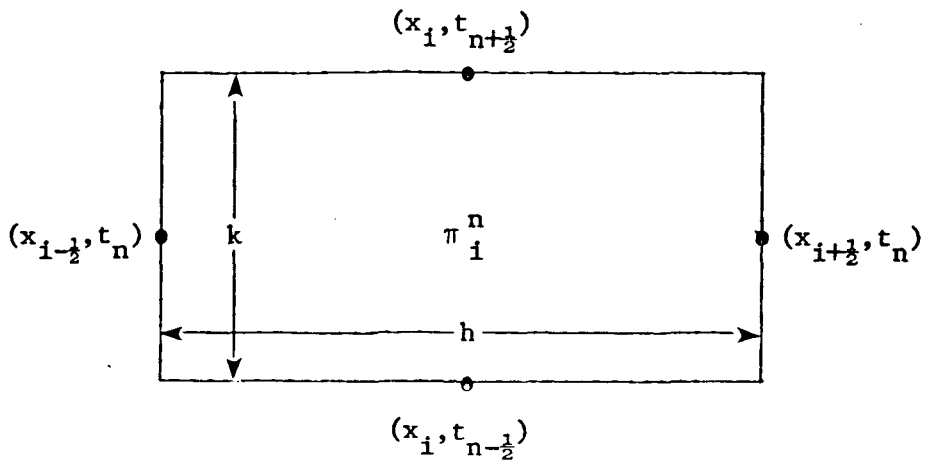

Figure 1: Points associated with the sides of a computational cell π_i^n.

Corresponding to (2.2) consider the leapfrog scheme

$$\begin{aligned} \overset{\bullet}{D}{}^h \rho &= 0 \\ \overset{\bullet}{D}{}^h(\rho u) + \delta_x p &= 0 \\ \overset{\bullet}{D}{}^h(\rho E) + \delta_x(pu) &= 0. \end{aligned} \tag{3.4}$$

The conservation property of (3.4) is expressed by the fact that if any of these equations is summed over any set of contiguous cells of Ω_h there remains only contributions arising from the boundary of the subdomain. These equations express three algebraic relationships between the twelve values of ρ, u, and E at the center points of the sides of the cell π_i^n. A mixed initial-boundary value problem for a cell may be posed in which ρ, u, E are jointly prescribed at $(x_i, t_{n-1/2})$ and, individually, at either boundary side $(x_{i\pm 1/2}, t_n)$. In order to algebraically determine ρ, u, E at $(x_i, t_{n+1/2})$ equations (3.2) must be supplemented by three further conditions.

In view of Lemma 1, if the conditions

$$(\mu_t\rho)(\mu_x u) = \mu_x(\rho u),$$

$$\mu_t u = \mu_x u, \tag{3.5}$$

$$\mu_t E = \mu_x E,$$

are adjoined to (3.4) a solution will also satisfy

$$\overset{\bullet h}{D}\rho = 0$$

$$(\mu_t\rho)D_t^h u + \delta_x p = 0 \tag{3.6}$$

$$(\mu_t\rho)D_t^h E + (\mu_x p)\delta_x u + (\mu_x u)\delta_x p = 0$$

which may be compared to (2.3).

From Lemma 2,

$$D_t^h E = D_t^h(\tfrac{1}{2}u^2+e) = (\mu_t u)D_t^h u + D_t^h e,$$

so that use of the momentum equation as expressed by the second equation in (3.6) yields the energy equation in the form

$$(\mu_t\rho)D_t^h e + (\mu_x p)\delta_x u = 0. \tag{3.7}$$

Referring to (2.4) the first law of thermodynamics may be written in the form

$$D_t^h e = T_h D_t^h s - (\mu_x p)D_t\rho^{-1}, \tag{3.8}$$

in which T_h is a temperature value interior to the cell π_i^n. From Lemma 1 as well as the definition of $\overset{\bullet h}{D}$ we have

$$\overset{\bullet h}{D}(\rho\rho^{-1}) = \delta_x u$$

$$= (\mu_t\rho)D_t^h\rho^{-1}.$$

Then (3.8) yields

$$(\mu_t\rho)D_t^h e + (\mu_x p)\delta_x u = (\mu_t\rho)T_h D_t^h s, \tag{3.9}$$

so that, in view of (3.7),

$$D_t^h s = 0. \tag{3.10}$$

These results are summarized in:

Theorem 1: If a mixed initial-boundary value problem for (3.4) is solvable in a cell π_i^n with the auxiliary conditions (3.5) the solution also satisfies (3.6) in which the conservation of energy equation can also be expressed by (3.7). Conversely, the solution of the problem formulated for (3.6) also satisfies (3.4). In either problem, $D_t^h s = 0$.

It is plausible to expect that the equivalence expressed by this theorem also will extend to the system of algebraic equations which results by treating a mixed initial-boundary value problem on the computational domain Ω_h which is composed of cells $\{\pi_i^n\}$. However, both the existence of a solution and its convergence as $h \to 0$ can only be expected to result when the formulation of the boundary conditions is consistent with inflow and outflow conditions which arise from the theory of characteristics of the differential equation (2.2) or (2.3). We shall not discuss this question here.

In the following section we shall describe a simple means of modifying the auxiliary conditions (3.5) so as to result in the entropy condition $D_t^h s \geq 0$.

4. An Entropy-Producing Scheme

When, instead of the auxiliary conditions (3.5), only the conditions

$$(\mu_t \rho)(\mu_x u) = \mu_x(\rho u)$$
$$\mu_t u = \phi_x u, \tag{4.1}$$

are adjoined to (3.4) the application of Lemma 1 results in

$$D^h \rho = 0$$
$$(\mu_t^h \rho) D_t u + \delta_x p = 0$$
$$(\mu_t \rho) D_t^h E + \delta_x(pu) = (\mu_t E - \mu_x E)\delta_x(\rho u),$$

instead of (3.6). Previous arguments show that the last equation may be replaced by

$$(\mu_t \rho) T_h D_t^h s = (\mu_t E - \mu_x E)\delta_x(\rho u). \tag{4.3}$$

Hence the condition

$$\mu_t E = \mu_x E + \sigma\delta_x(\rho u), \qquad \sigma > 0, \tag{4.4}$$

will result in

$$D_t^h s \geq 0,$$

and thus will yield an entropy-producing scheme when combined with (4.1).

In the application of this theory a simplified approximation to conditions (3.5) or to (4.1) and (4.4) is useful. We have

Lemma 3: Identically,

$$\mu_x(\phi\psi) \equiv (\mu_x\phi)(\mu_x\psi) + \tfrac{1}{2}(\Delta x)^2\delta_x\phi\delta_x\psi,$$
$$\mu_t(\phi\psi) \equiv (\phi_t\phi)(\mu_t\psi) + \tfrac{1}{2}(\Delta t)^2\delta_t\phi\delta_t\psi.$$

Hence, if the divided difference quotients involved remain bounded as $h \to 0$,

$$\mu(\phi\psi) = (\mu\phi)(\mu\psi) + O(h^2).$$

If $\mu_t u = \mu_x u$ an application of this lemma shows that

$$\mu(u^2) = (\mu u)^2 + O(h^2),$$

so that

$$(\mu_t E - \mu_x E) = (\mu_t e - \mu_x e) + O(h^2).$$

Also,

$$(\mu_t\rho)(\mu_x u) - \mu_x(\rho u) = (\mu_t\rho - \mu_x\rho)(\mu_x u) + O(h^2).$$

Then, to terms of order h^2, condition (4.1) may be approximated by

$$\begin{aligned} \mu_t\rho &= \mu_x\rho \\ \mu_t u &= \mu_x u, \end{aligned} \tag{4.5}$$

and condition (4.4) by

$$\mu_t e = \mu_x e + \sigma\delta_x(\rho u). \tag{4.6}$$

With these approximations the argument which allowed the energy equation in (3.6) to be expressed in terms of e by (3.7) also yields (4.2) in the form

$$D_t^h \rho = 0$$

$$(\mu_x \rho) D_t^h u + \delta_x u = 0 \tag{4.7}$$

$$(\mu_x \rho) D_t^h e + (\mu_x p)(\delta_x u) = \sigma \left(\delta_x (\rho u)\right)^2$$

$$= (\mu_x \rho) T_h D_t^h s \geq 0.$$

In order to avoid having the $0(h^2)$ approximations used in (4.5) and (4.6) destroy the entropy condition $D_t^h s \geq 0$ we may take

$$\sigma = \sigma_0 h + 0(h^2). \tag{4.8}$$

We summarize these results in the following

Theorem 2: Under the auxiliary conditions (4.1) and (4.4) the leapfrog scheme (3.4) is entropy producing and is equivalent to (4.2). This equivalence is preserved to terms of second order in h when (4.1) and (4.4) are replaced by (4.5) and (4.6) and when (4.2) is replaced by (4.7).

The leapfrog scheme (3.4) expresses, to terms of second order in Δx and Δt, the integral form of the differential equations (2.2) in a cell π_i^n when the values occurring in (3.4) are interpreted as average values along the sides of the cell. If we regard as equivalent any two solutions of the differential equations (2.2) in a cell which have the same average values of their initial and boundary data then any equivalent solutions of (2.2) will satisfy (3.4) to the same degree of approximation. This equivalence class includes solutions of (2.2) which have discontinuities interior to π_i^n. It is not difficult to conclude that if a solution of (3.4), (4.1), and (4.4) converges in Ω_h for $h \to 0$ in the sense of bounded L_1^{loc} convergence then the

solution will also satisfy the integral form of (2.2) on every subdomain of Ω and moreover, will lead to a weak form of the entropy inequality $D_t s \geq 0$.

The discussion in Part I has established the equivalent treatment of conservation and nonconservation forms of the Euler equations by compact finite difference schemes. A discussion of numerical methods for treating such schemes has been given in Philips and Rose [4]. Although the scheme employed in that paper differs from the scheme described here in the manner in which dissipation is introduced the difference is slight and their numerical results for a Riemann problem also serve to validate the methods described here.

We ask the reader to verify that no essential change in our treatment of the one-dimensional case is required in order to extend our argument to higher space dimensions.

Part II

II.1. Introduction

The Euler equations arise as the formal singular perturbation limit of the Navier-Stokes equations as the coefficients of viscosity and heat conduction vanish and it may be conjectured that the class of correct mathematical boundary conditions for the Euler equations are determined by the "outer expansions" of the Navier-Stokes equations in the sense of singular perturbation theory. A simple energy argument suggests that the Navier-Stokes equations are well-posed under boundary conditions which are independent of the Mach number (Problem P). In contrast, the theory of characteristics for hyperbolic equations shows that the number of boundary conditions for the Euler equations depends upon the Mach number and is, generally, less than the number of boundary conditions which are appropriate for the Navier-Stokes equations. This reduction in the number of boundary conditions is a characteristic feature of singular perturbation problems.

Motivated by Part I we here describe a compact finite difference scheme for treating the compressible Navier-Stokes equations. A study of the resulting finite-difference scheme (3.9) shows that only the Euler boundary conditions have an appreciable effect on the solution as the viscosity and heat conduction coefficients vanish. As a result the difference scheme (3.9) provides for a treatment of both problems under boundary conditions which are independent of the Mach number. This can be important in practical problems where characteristics may not be known a priori.

II.2. The Navier-Stokes Equations

Let k, μ, λ, c_v, c_p denote the coefficient of heat conduction, the shear and second coefficients of viscosity, and the specific heats at

constant volume and pressure, respectively. If $R = c_p - c_v$ so that $\gamma = c_p/c_v$ the equation of state in terms of the temperature T may be written $p = R\rho T$.

In terms of the operator $\dot{D}$ defined in Part I (eq. (2.1)) the conservation form of the Navier-Stokes equations is given as

$$\begin{aligned} \dot{D}_\rho &= 0 \\ \dot{D}_\rho \underline{u} + \operatorname{grad} p &= \operatorname{div} \pi' \\ \dot{D}_\rho E + \operatorname{div}(p\underline{u}) &= \operatorname{div}(\underline{u}\pi') - \operatorname{div} \underline{h} \end{aligned} \tag{2.1}$$

in which π' is the reduced stress tensor

$$\pi' \equiv \begin{pmatrix} 2\mu u_x + \lambda \operatorname{div} \underline{u} & \mu(u_y + v_x) \\ \mu(u_y + v_x) & 2\mu v_y + \lambda \operatorname{div} \underline{u} \end{pmatrix} \tag{2.2}$$

and $\underline{h}$ is the heat flux given as

$$\underline{h} = -k \operatorname{grad} T \tag{2.3}$$

The relationship between the operators $\dot{D}$ and D_t described in Part I may be used to transform (2.1) to the nonconservation form

$$\begin{aligned} \dot{D}_\rho &= 0 \\ D_t \underline{u} + R \operatorname{grad} T + \rho^{-1} RT \operatorname{grad} \rho &= \rho^{-1} \operatorname{div} \pi' \\ D_t T + (\gamma - 1) T \operatorname{div} \underline{u} &= (\rho c_v)^{-1} (\pi' \operatorname{grad} \underline{u} - \operatorname{div} \underline{h}). \end{aligned} \tag{2.4}$$

If

$$f \equiv \pi' \operatorname{grad} \underline{u} \tag{2.5}$$

it is not difficult to verify that

(2.6) $$f = \mu(\rho c_v)^{-1}[u_x^2 + v_y^2 + 2(u_y + v_x)^2 + \lambda(\text{div } \underline{u})^2]$$

is nonnegative. Hence, using (2.3), the energy equation in (2.4) assumes the form

(2.7) $$D_t T + (\gamma-1)T \text{ div } \underline{u} = k(\rho c_v)^{-1} \text{ div grad } T + f$$

Using the thermodynamic relationship given by (2.4) in Part I it is a simple exercise to verify that the second law of thermodynamics holds in the form

(2.8) $$\rho D_t s + \text{div } (T^{-1} \underline{h}) \geq 0.$$

The equations (2.4) and (2.7) may then be expressed in the form

(2.9) $$U_t + A U_x + B U_y = C U_{xx} + 2D U_{xy} + E U_{yy} + F$$

where the transpose of U is (ρ, u, v, T) and

$$A = \left(\begin{array}{c|ccc} u & \rho & 0 & 0 \\ \hline RT/\rho & u & 0 & R \\ 0 & \hat{0} & u & 0 \\ 0 & (\gamma-1)T & 0 & u \end{array}\right) = \left(\begin{array}{c|c} a_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$$

$$B = \left(\begin{array}{c|ccc} v & 0 & \rho & 0 \\ \hline 0 & v & 0 & 0 \\ RT/\rho & 0 & v & R \\ 0 & 0 & (\gamma-1)T & v \end{array}\right) = \left(\begin{array}{c|c} b_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$$

$$C = \mu\rho^{-1}\left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 2+\lambda/\mu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma/P_r \end{array}\right) = \mu\rho^{-1}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & C_{22} \end{array}\right)$$

(2.10)

$$D = \frac{\mu\rho^{-1}}{2}\left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1+\lambda/\mu & 0 \\ 0 & 1+\lambda/\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \mu\rho^{-1}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_{22} \end{array}\right)$$

$$E = \mu\rho^{-1}\left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 2+\lambda/\mu & 0 \\ 0 & 0 & 0 & \gamma/P_r \end{array}\right) = \mu\rho^{-1}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & E_{22} \end{array}\right)$$

$$F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f \end{pmatrix},$$

in which $P_r = \frac{\mu c_p}{k}$ is the Prandtl number.

It will be convenient to introduce the matrix

$$J = \mathrm{diag}(0,1,1,1), \tag{2.11}$$

and to write equations (2.9) in system form as

$$\begin{aligned} U_t + AU_x + BU_y &= J(V_x + W_y + F) \\ CU_x + DU_y &= JV \\ DU_x + EU_y &= JW. \end{aligned} \tag{2.12}$$

Because the elliptic operator on the right-hand side of equation (2.12) has rank 3, it is not immediately apparent how boundary conditions may be imposed. To this end, consider the one-dimensional problem

$$U_t + AU_x = CU_{xx}, \tag{2.13}$$

where A is symmetric, and constant,

$$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix},$$

and $a_{11} > 0$ while

$$C = \begin{pmatrix} 0 & 0 \\ 0 & C_{22} \end{pmatrix},$$

is non-negative. With initial and boundary conditions given by

$$\text{a)} \quad U(x,0) = \bar{U} \,,$$

(2.14) $$\text{b)} \quad U(0,t) = 0 \,,$$

$$\text{c)} \quad JU(1,t) = 0 \,,$$

if (2.13) is multiplied by U^T and then integrated the result is the "energy" expression

$$0 = \tfrac{1}{2} \frac{d}{dt} \int_0^1 U^T U dx + \int_0^1 U_x^T C U_x dx + U^T (\tfrac{1}{2} AU - CU_x) \Big|_0^1 .$$

Employing the initial and boundary conditions (2.14) and noting that $a_{11} > 0$ by assumption, there results

(2.15) $$\int_0^1 U^T(x,t) U(x,t) dx \leq \int_0^1 \bar{U}^T(x) \bar{U}(x) dx,$$

where the equality applies if and only if $U_x = \text{const}$. This, of course, implies the uniqueness of the solution for the linear problem considered.

Were this argument applicable to the hydrodynamic problem (2.9) we would have $a_{11} = u$ in which case the boundary conditions (2.14b) and (2.14c) would correspond to inflow and outflow conditions. We thus state:

Problem P: Solve the Navier-Stokes equations in the form (2.12) in a domain $\mathcal{D}$ under the initial and boundary conditions

$$U(x,y,0) = \bar{U}$$

(2.16) $$U(\cdot,t) = U \text{ inflow}$$

$$JU(\cdot,t) = JU \text{ outflow}.$$

More specifically, we assume $\mathcal{D}$ is the unit square on which inflow conditions apply for $x=0$ or $y=0$ while outflow conditions apply for $x=1$ or $y=1$.

A more complete discussion of properly-posed boundary conditions for problems of this type has been given by Strikwerda [6].

II.3. A Compact Finite Difference Scheme

If we ignore for the time being the fact that the term F in (2.9) is a function of U_x and U_y this equation is similar to the type of problem which was treated by Philips and Rose [4] by means of a second-order accurate compact finite difference scheme. However, their argument depended essentially upon the fact that the coefficient matrices C, D, E in (2.9) were nonsingular; in order to describe the extension necessary when these coefficients are singular (compare (2.10)) it appears simplest to rederive the derivation of the difference equations from elementary principles. This is done here.

We suppose the computational domain can be subdivided into rectangular computational cells $\pi^n_{jk}\{(x,y,t): |x-x_j| < \Delta x/2,\ |y-y_k| < \Delta y/2, |t-t_n| < \frac{\Delta t}{2}\}$. Again, write $U^n_{jk} = U(j\Delta x, k\Delta y, n\Delta t)$ and employ the notation

$$\mu_x U^n_{jk} = (U^n_{j+\frac{1}{2},k} + U^n_{j-\frac{1}{2},k})/2 \ , \tag{3.1}$$

$$\delta_x U^n_{jk} = (U^n_{j+\frac{1}{2},k} - U^n_{j-\frac{1}{2},k})/\Delta x \ ,$$

etc. When no confusion is likely to arise we suppress the spatial indices by writing $U^n = U(\cdot,\cdot,n\Delta t)$; thus $\mu_x U^n$, $\delta_x U^n$, $\mu_y U^n$, $\delta_y U^n$, $\mu_t U^n$, $\delta_t U^n$ involve the values of U at the center points of the faces of the cell π^n.

The approximation method to be described is based upon the following idea: suppose the solution $U = U^*$ of (2.11) is known to be smooth; then the result of approximating the coefficient matrices in (2.11) by their values averaged over each computational cell π^n, say $A^n = A^n(U^*)$, etc., leads to a linear partial differential equation in each cell

$$
\begin{aligned}
U_t + A^n U_x + B^n U_y &= J(V_x + W_y + F^n), \\
C^n U_x + D^n U_y &= J\, V_x, \qquad (3.2) \\
D^n U_x + E^n U_y &= J\, W_x.
\end{aligned}
$$

This system will approximate (2.11) to terms of second-order in the mesh parameters if π^n is sufficiently small. Because (3.2) is linear it is feasible to construct a linear manifold of solutions in each cell and then, by means of algebraic equations which express continuity conditions at the boundaries of neighboring cells together with the initial and boundary conditions associated with the problem, determine a specific manifold which leads to an approximation to the solution of (2.11). These algebraic conditions are expressed by the finite difference equations (3.9) whose development we now describe.

The following discussion concerns (3.2) in a fixed cell π^n. With the coefficient matrices partitioned as in (2.10) introduce the following

Definitions: I_3 is the 3×3 identity matrix; for $a_{11} \neq 0$, $b_{11} \neq 0$,

$$\tilde{A} \equiv \begin{pmatrix} 0 & -a_{11}^{-1}A_{12} \\ 0 & I_3 \end{pmatrix}, \qquad \tilde{B} \equiv \begin{pmatrix} 0 & -b_{11}^{-1}B_{12} \\ 0 & I_3 \end{pmatrix},$$

$$\tilde{A}_{22} \equiv A_{22} - A_{21}a_{11}^{-1}A_{12}, \qquad \tilde{B}_{22} \equiv B_{22} - B_{21}b_{11}^{-1}B_{12},$$

$$\omega_x \equiv C_{22}^{-1}\tilde{A}_{22}, \qquad \theta_x \equiv \frac{\Delta x}{2}\omega_x,$$

(3.3)

$$\omega_y \equiv E_{22}^{-1}\tilde{B}_{22}, \qquad \theta_y \equiv \frac{\Delta x}{2}\omega_y,$$

$$\Omega(\omega x) \equiv \begin{pmatrix} 1 & 0 \\ 0 & \exp \omega x \end{pmatrix}, \qquad [C]^{-1} = \rho\mu^{-1}\begin{pmatrix} 0 & 0 \\ 0 & C_{22}^{-1} \end{pmatrix}.$$

Note that the system of differential equations $AY = CY'$ has the general solution $Y(x) = \tilde{A}\Omega(x\omega_x)\underline{\alpha}$ where $\underline{\alpha}$ is a vector parameter.

Each of the terms I, $(xI - tA)$, $(yI - tB)$, $\tilde{A}\Omega(x\omega_x)$, $\tilde{B}\Omega(y\omega_y)$ is thus a solution of (3.2) when $F = 0$ so that

$$U = \underline{\alpha}_1 + (xI - tA)\underline{\alpha}_2 + (yI - tB)\underline{\alpha}_3 + \tilde{A}\Omega(x\omega_x)\underline{\alpha}_4 + \tilde{B}\Omega(y\omega_y)\underline{\alpha}_5 + tJF, \tag{3.4}$$

describes a solution manifold of (3.2).

Introduce the definitions

$$q(\theta) \equiv \coth\theta - \theta^{-1},$$

$$r(\theta) \equiv \theta^{-1} - (\sinh\theta)^{-1} \tag{3.8}$$

and consider the compact finite difference scheme

$$\text{a)}\quad (\delta_t + A\delta_x + B\delta_y)U^n = J(\delta_x V^n + \delta_y W^n + F^n)$$

$$\text{b)}\quad \mu_t U^n = \mu_x U^n = \mu_y U^n$$

$$\text{c)}\quad (C\delta_x + D\delta_y)U^n = (J\mu_x - \frac{\Delta x}{2} Q_x\cdot\delta_x)V^n + \frac{\Delta y}{2} R_y\delta_y W^n \tag{3.9}$$

$$\text{d)}\quad (D\delta_x + E\delta_y)U^n = (J\mu_y - \frac{\Delta y}{2} Q_y\cdot\delta_y)W^n + \frac{\Delta x}{2} R_x\delta_x V^n ,$$

in which

$$Q_x \equiv \begin{pmatrix} 0 & 0 \\ 0 & C_{22}q(\theta_x)C_{22}^{-1} \end{pmatrix},$$

$$Q_y \equiv \begin{pmatrix} 0 & 0 \\ 0 & E_{22}q(\theta_y)E_{22}^{-1} \end{pmatrix},$$

(3.10)

$$R_x \equiv \begin{pmatrix} 0 & 0 \\ 0 & D_{22}r(\theta_x)C_{22}^{-1} \end{pmatrix},$$

$$R_y \equiv \begin{pmatrix} 0 & 0 \\ 0 & D_{22}r(\theta_y)E_{22}^{-1} \end{pmatrix},$$

where θ_x, θ_y are defined by (3.3).

It is not difficult to verify that the solutions $I, (xI - tA), (yI - tB)$ of (3.2) with $F = 0$ satisfy (3.9), i.e., the truncation error resulting from these solutions of (3.2) vanishes when $F = 0$. More complex algebraic manipulations are required to verify that, as a result of the definitions of Q and R given by (3.10), the truncation error due to the solutions $\tilde{A}\Omega(x\omega_x)$ and $\tilde{B}\Omega(y\omega_y)$ of (3.2) when $F = 0$ also vanish. Thus, the difference equations (3.9) result in a zero truncation error when applied to the solution manifold (3.4). (This argument is simpler than that employed by Philips and Rose [4] and helps illuminate the role of the matrices Q and R in (3.9) in reducing the truncation error on the solution manifold.)

For real values of θ the functions $q(\theta)$ and $r(\theta)$ given by (3.8) are regular in θ and are conveniently evaluated by

(3.11) (a)
$$q(\theta) \simeq \theta/3 \quad , \quad \theta \text{ small}$$
$$\simeq \operatorname{sgn}\theta \quad , \quad \theta \text{ large}$$

where $\operatorname{sgn}\theta = \theta/|\theta|$; also

(b)
$$r(\theta) \simeq \theta/6 \quad , \quad \theta \text{ small}$$
$$\simeq \theta^{-1} \quad , \quad \theta \text{ large.}$$

The matrices θ_x, θ_y given by (3.3) are generalizations of the cell Reynolds number. Consider θ_x: if S is the matrix which diagonalizes θ_x, say $S^{-1}\theta_x S = \tilde{\theta}_x$, then

(3.12)
$$q(\theta_x) = S\, q(\tilde{\theta}_x) S^{-1},$$
$$r(\theta_x) = S\, r(\tilde{\theta}_x) S^{-1},$$

and the approximations given in (3.11) may be used to evaluate $q(\tilde{\theta}_x)$, $r(\tilde{\theta}_x)$.

As mentioned earlier, the difference equations (3.9) generalize similar equations which were described in Philips and Rose [4] when the matrices C, D, and E were nonsingular. Arguments given there may be used to show that the truncation error in (3.9) is second order in the mesh parameters independent of θ_x, θ_y.

The reader is asked to verify the fact that the algebraic equations expressed by (3.9) together with (2.9) lead to a determined system of equations for U^n, V^n, and W^n. When the coefficient matrices in (3.9) are symmetric and constant an energy-norm estimate for the solution may be given (cf. [4]); in that case the existence and uniqueness of the solution and also the convergence of the scheme for any fixed values of the mesh parameters $\lambda_x = \Delta t/\Delta x$, $\lambda_y = \Delta t/\Delta y$ results. It is plausible that similar results hold when the coefficient matrices in (3.9) are variable and we appeal to this plausibility argument in the following discussion without explicit comment.

II.4. Solution Methods

a) As described in [4], compact schemes of the type (3.9) may be solved by a

two-step method:

i) by eliminating the value $U^{n+\frac{1}{2}}$ common to (3.9a) and (3.9b) there results, with $\tau = \Delta t/2$,

$$P_x \begin{pmatrix} U^n \\ V^n \end{pmatrix} + \tau R_y \begin{pmatrix} U^n \\ W^n \end{pmatrix} = \begin{pmatrix} U^{n-\frac{1}{2}} \\ 0 \end{pmatrix} + \tau F^n \quad ,$$

(4.1)

$$P_y \begin{pmatrix} U^n \\ W^n \end{pmatrix} + \tau R_x \begin{pmatrix} U^n \\ V^n \end{pmatrix} = \begin{pmatrix} U^{n-\frac{1}{2}} \\ 0 \end{pmatrix} + \tau F^n$$

where

$$P_x \equiv \begin{pmatrix} \mu_x + \tau A\delta_x & -\tau\delta_x J \\ C\delta_x & \frac{\Delta x}{2} Q_x\delta_x - \mu_x J \end{pmatrix} ,$$

$$P_y \equiv \begin{pmatrix} \mu_y + \tau B\delta_y & -\tau\delta_y J \\ E\delta_y & \frac{\Delta x}{2} Q_y\delta_y - \mu_y J \end{pmatrix} , \tag{4.2}$$

$$R_x \equiv \begin{pmatrix} A\delta_x & -\delta_x J \\ \tau^{-1}D\delta_x & -\tau^{-1}R(\theta_x)\delta_x \end{pmatrix} ,$$

$$R_y \equiv \begin{pmatrix} B\delta_y & -\delta_y J \\ \tau^{-1}D\delta_y & -\tau^{-1}R(\theta_y)\delta_y \end{pmatrix} .$$

The solution of (4.2) is determined by $U^{n-\frac{1}{2}}$ and the imposed boundary conditions for U^n. A formal ADI solution of (4.1), accurate to $0(\tau^2)$, is given by

$$\begin{pmatrix} U^n \\ V^n \end{pmatrix} = P_x^{-1}(I - \tau R_y P_y^{-1}) \left[\begin{pmatrix} U^{n-\frac{1}{2}} \\ 0 \end{pmatrix} + \tau F^n \right] ,$$

$$\begin{pmatrix} U^n \\ W^n \end{pmatrix} = P_y^{-1}(I - \tau R_x P_x^{-1}) \left[\begin{pmatrix} U^{n-\frac{1}{2}} \\ 0 \end{pmatrix} + \tau F^n \right] . \tag{4.3}$$

(ii) Using the solution U^n, V^n, W^n obtained from (4.1) $U^{n+\frac{1}{2}}$ may be calculated from either the "leapfrog" equation (3.9a) or from (3.9b).

In employing (4.1) the coefficient matrices are assumed to be evaluated at the center point of the cell π^n by spatial averages in the cell. We shall not pause to indicate how this may be approximated in the solution algorithm.

A drawback in employing (4.3) to solve (4.1) is that Δt must be suitably restricted; when the viscosity μ in (2.3) is sufficiently small this restriction is approximated by the CFL condition for the dominant hyperbolic part of the operator in (2.2). Presumably, in view of earlier remarks, (4.1) is solvable for any value of the ratio of mesh parameters λ_x, λ_y. In order to exploit this, particularly for the calculation of steady-state solutions of (3.9), a more effective solution method than (4.3) is required. This topic will not be treated here.

We remark, finally, that the existence of the unique solution of the algebraic equations (4.1) is a consequence of the (assumed) existence and uniqueness of the finite difference equations (3.9).

b) The operators P_x, P_y in (4.3) involve the solution of algebraic two-point boundary value problems which can be obtained by a method due to Keller [1]. A simpler solution method results by observing that U^n may be obtained directly by solving a block tridiagonal system of equations (cf. [3]) as will now be shown. The asymptotic consequences when $\mu \to 0$ will be described in section 6.

The solution of the one-dimensional example

$$P_x \begin{pmatrix} U_j^n \\ V_j^n \end{pmatrix} = \begin{pmatrix} g_{1,j} \\ g_{2,j} \end{pmatrix} ,$$

typifies the problem involved in applying (4.3) where P_x is given by (4.2). In a cell π_j^n the first equation in this system can be written

$$J'(\mu_x + \tau A\delta_x)\, U = J' g_1$$

where $J' \equiv I - J$. The remaining equations can be solved for the values $V^n_{j\pm\frac{1}{2}}$ with the result

$$\begin{aligned} \lambda_x V^n_{j+\frac{1}{2}} &= J(a_j^+ U^n_{j+\frac{1}{2}} + b_j^+ U^n_{j-\frac{1}{2}} - g_j^+) \ , \\ \lambda_x V^n_{j-\frac{1}{2}} &= J(a_j^- U^n_{j+\frac{1}{2}} + b_j^- U^n_{j-\frac{1}{2}} + g_j^-) \ , \end{aligned} \tag{4.5}$$

in which

$$\begin{aligned} \lambda_x &= \Delta t/\Delta x, \qquad \kappa_x = 2\lambda x/\Delta x \ , \\ a^{\pm} &= \tfrac{1}{2}[(Q_x \pm I)(I + \lambda_x A) + \kappa_x C], \\ b^{\pm} &= \tfrac{1}{2}[(Q_x \pm I)(I - \lambda_x A) - \kappa_x C], \end{aligned} \tag{4.6}$$

and

$$g^{\pm} = [(I \pm Q_x) g_1 \pm \lambda_x g_2].$$

The pair of values $U^n_{j+\frac{1}{2}}$, $V^n_{j+\frac{1}{2}}$ are common to the contiguous cells π_j^n, π_{j+1}^n. Expressions for the value $V^n_{j+\frac{1}{2}}$ in each such cell are given by (4.5); the result of equating these expressions for $V^n_{j+\frac{1}{2}}$ and setting $\ell = j+\frac{1}{2}$ is

$$-Ja^-_{\ell+\frac{1}{2}} U^n_{\ell+1} + b^+_{\ell-\frac{1}{2}} U^n_{\ell-1} + (a^+_{\ell-\frac{1}{2}} - Jb^-_{\ell+\frac{1}{2}}) U^n_{\ell} = g^+_{\ell-\frac{1}{2}} + Jg^-_{\ell+\frac{1}{2}} \ . \tag{4.7}$$

This block-tridiagonal system of equations may be efficiently solved for U^n with the boundary conditions prescribed by (2.9) and the values V^n

can then be obtained from (4.5). However, in order to evaluate the coefficient matrices $a^{\pm}$, $b^{\pm}$ an effective means of approximating the matrix Q_x defined by (3.10) must be considered. This topic, with specific reference to the Navier-Stokes equations (2.2) is the subject of the next section.

II.5. The Matrices Q and R

The matrix Q_x occurring in the coefficients a, b in (4.6) was defined in terms of A and C by (3.10) in terms of the matrix $q(\theta_x)$, which itself was defined by (3.3) and (3.8). The matrix Q_y is similarly defined in terms of the matrices B and E.

Confining our attention to Q_x, first note that Q_x is given, using (3.3), by

$$\theta_x = \frac{\rho\Delta x}{2\mu}\begin{pmatrix} (u - RT/u)\delta & 0 & R\delta \\ 0 & u & 0 \\ \varepsilon(\gamma-1)T & 0 & \varepsilon u \end{pmatrix}, \tag{5.1}$$

in which

$$\delta = (2+\lambda/\mu)^{-1}, \qquad \varepsilon = P_r/\gamma \quad . \tag{5.2}$$

Denote the eigenvalues of $(\frac{2\mu}{\rho\Delta x})\theta_x$ by $\theta_{x,1}, \theta_{x,2}, \theta_{x,3}$. If

$$\sigma = \varepsilon/\delta, \tag{5.3}$$

then

$$\theta_{x,1} = u, \tag{5.4}$$

and $\theta_{x,2}, \theta_{x,3}$ are given as

$$(5.5) \qquad \frac{2\theta_{x,\ell}}{u\delta} = \left(1+\sigma - \frac{1}{\gamma M^2}\right) + (-1)^{\ell} M^{-1}\left[M^2\left((1+\sigma) - \frac{1}{\gamma M^2}\right)^2 + 4\sigma(1-M^2)\right]^{\frac{1}{2}}, \quad \ell = 2,3,$$

where $M = u/c$, $c^2 = \gamma RT$. The following approximations result:

M = 1

$$(5.6) \qquad \begin{aligned} \frac{2\theta_{x,2}}{u\delta} &= 2\left(1+\sigma - \frac{1}{\gamma}\right) > 0, \\ \frac{2\theta_{x,3}}{u\delta} &= 0 . \end{aligned}$$

M small

$$(5.7) \qquad \begin{aligned} \frac{2\theta_{x,2}}{u\delta} &\simeq 1 + \sigma > 0, \\ \frac{2\theta_{x,3}}{u\delta} &\simeq (1+\sigma) - \frac{2}{\gamma M^2} < 0. \end{aligned}$$

M large

$$(5.8) \qquad \begin{aligned} \frac{\theta_{x,2}}{u\delta} &\simeq 1, \\ \frac{\theta_{x,3}}{u\delta} &\simeq \sigma. \end{aligned}$$

Writing

$$(5.9) \qquad \hat{\theta}_x = \begin{pmatrix} \theta_{x,1} & 0 & 0 \\ 0 & \theta_{x,2} & 0 \\ 0 & 0 & \theta_{x,3} \end{pmatrix} ,$$

then

$$(5.10) \qquad \theta_x = \frac{\rho\Delta x}{2\mu} S \cdot \hat{\theta}_x \cdot S^{-1} ,$$

where

$$S = \begin{pmatrix} 0 & s_2 & s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} ,$$

(5.11)

$$S^{-1} = \begin{pmatrix} 0 & s_3 - s_2 & 0 \\ -1 & 0 & s_3 \\ 1 & 0 & -s_2 \end{pmatrix} \div (s_3 - s_2) ,$$

in which

(5.12) $$s_\nu = \frac{(\theta_{x,\nu} - \varepsilon u)}{\varepsilon(\gamma-1)T} , \qquad \nu = 2,3.$$

As a result, using (3.12)

(5.13) $$q(\theta_x) = S\, q\left(\left(\frac{\rho\Delta x}{2\mu}\right) \hat{\theta}_x\right) S^{-1} ,$$

in which $q\left(\left(\frac{\rho\Delta x}{2\mu}\right)\hat{\theta}_x\right)$ may be approximated by using (3.8).

In view of (5.6), (5.7), (5.8),

(5.14) $$\lim_{\mu \to 0} q(\theta_x) = S\left(\text{sgn } u \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi_x \end{pmatrix}\right) S^{-1} ,$$

where

$$\xi_x = -1 , \quad M_x < 1$$
$$= 0 , \quad M_x = 0$$
$$= 1 , \quad M_x > 1.$$

where $M_x = u/c$.

Thus,

$$(5.16) \qquad Q_x = \begin{pmatrix} 0 & 0 \\ 0 & C_{22}S\left[\text{diag}\left(q(\theta_{x,1}), q(\theta_{x,2}), q(\theta_{x,3})\right)\right]S^{-1}C_{22}^{-1} \end{pmatrix},$$

and

$$(5.17) \lim_{\mu \to 0} Q_x = \begin{pmatrix} 0 & 0 \\ 0 & C_{22}S\left[\text{diag}(1,1,\xi_x)\right]S^{-1} \operatorname{sgn} u \cdot C_{22}^{-1} \end{pmatrix}.$$

Similar expressions result for Q_y noting (3.10).

In the same manner, using (3.8) and (3.10),

$$(5.18) \qquad R_x = \begin{pmatrix} 0 & 0 \\ 0 & D_{22}S\left[\text{diag}\left(r(\theta_{x,1})r(\theta_{x,2})r(\theta_{x,3})\right)\right]S^{-1}\tilde{A}_{22}^{-1} \end{pmatrix},$$

$$(5.19)\qquad \lim_{\mu \to 0} R_x = \begin{pmatrix} 0 & 0 \\ 0 & D_{22} S \left[\operatorname{diag}(1,1,1) \right] S^{-1} \tilde{A}_{22} \end{pmatrix}$$

with similar results for R_y.

Using (3.8) and (3.11), the results of this section allow the coefficient matrices Q and R in (3.9) to be evaluated as well as the coefficient matrices in (4.7) as described by (4.6).

II.6. The Euler Equations

Assume that $\lambda = k = 0$. The Euler equations

$$(6.1)\qquad U_t + A U_x + B U_y = 0,$$

then arise as the formal limit of the Navier-Stokes equations (2.12) as the viscosity $\mu \to 0$. If $U(\mu)$ denotes the solution of the Navier-Stokes equations with certain initial and boundary conditions, singular perturbation methods provide an important means of describing the sense in which $U(\mu)$ may be approximated by a solution U of the Euler equations (6.1) in regions exterior to boundary layers, shocks, etc. where vorticity can be generated.

The solution $U^n(\mu)$ of the finite difference equations (3.9) together with (3.4) determines an approximate solution, say $U(\mu,\Delta x)$, of $U(\mu)$ if we assume that $U(\mu,\Delta x) \to U(\mu)$ as $\Delta x \to 0$. The construction employed in (3.4) is similar in viewpoint to one which could be employed by a singular perturbation method if one were to allow a much greater degree of algebraic complexity to be used in order to impose connection formulas between subdomains than is practical when analytic results are primarily desired. If, formally, $\lim_{\mu \to 0} U(\mu,\Delta x) = U(\Delta x)$ it is thus reasonable to conjecture that $U(\Delta x)$ provides an approximation to the Euler solution U as well.

An important mathematical difference between the Navier-Stokes equations (2.12) and the Euler equation (6.1) lies in the formulation of boundary conditions. For (2.12) $U(\mu)$ may be prescribed at boundaries as indicated by (2.9) while for (6.1) only certain combinations of U as determined by characteristics are permissible. This reduction of boundary conditions is, of course, a familiar feature of singular perturbation problems.

We now propose to examine how the Euler boundary conditions result from $U(\mu,\Delta x)$ when $\mu \to 0$ when (3.9) is employed.

As described in II.4, the ADI solution method (4.3) used to solve (3.9) can be effectively solved by employing the block-tridiagonal system (4.7) which we now consider in the simplified form

$$-Ja^{-}\, U^{n}_{\ell+1} + b^{+}\, U^{n}_{\ell-1} + c\, U^{n}_{\ell} = g_{\ell}, \qquad \ell = 1,2,\ldots,L-1, \tag{6.2}$$

where here U^{n}_{0} and JU^{n}_{L} are prescribed as inflow and outflow conditions (cf. (2.16)).

With S given by (5.11), let

$$\hat{Q}_x = \begin{pmatrix} 0 & 0 \\ 0 & S[\mathrm{diag}(1,1,\xi_x)]S^{-1}\ \mathrm{sgn}\, u \end{pmatrix} ; \tag{6.3}$$

using (3.3) to define $[C]^{-1}$, (5.17) may be written

$$\lim_{\mu \to 0} Q_x = C\, \hat{Q}_x [C]^{-1}, \tag{6.4}$$

while, according to (4.6),

$$\begin{aligned} \lim_{\mu \to 0} Ja^{-} &= \tfrac{1}{2}\left[C(\hat{Q}_x - J)[C]^{-1}(I + \lambda_x A)\right], \\ \lim_{\mu \to 0} b^{+} &= \tfrac{1}{2}\left[\left((I - J) + C(\hat{Q}_x + J)[C]^{-1}\right)(I + \lambda_x A)\right]. \end{aligned} \tag{6.5}$$

A simple calculation yields

$$(6.6)\qquad S[\mathrm{diag}(1,1,\xi_x)]S^{-1} = (\delta s)^{-1}\begin{pmatrix} \xi_x s_3 - s_2 & 0 & (1-\xi_x)s_2 s_3 \\ 0 & \delta s & 0 \\ \xi_x - 1 & 0 & s_3 - \xi_x s_2 \end{pmatrix},$$

where $\delta s = s_3 - s_2$. Thus, assuming $u > 0$, $\operatorname{sgn} u = 1$ so that

$$(6.7)\qquad \hat{Q}_x - J = (\delta s)^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (\xi_x - 1)s_3 & 0 & (1-\xi_x)s_2 s_3 \\ 0 & 0 & 0 & 0 \\ 0 & (\xi_x - 1) & 0 & (1-\xi_x)s_2 \end{pmatrix},$$

$$(6.8)\qquad \hat{Q}_x + J = (\delta s)^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (\xi_x+1)s_3+2s_2 & 0 & (1-\xi_x)s\ s_3 \\ 0 & 0 & 2\delta s & 0 \\ 0 & (\xi_x-1) & 0 & 2s_3-(\xi_x+1)s_2 \end{pmatrix}.$$

Suppose $\mu \to 0$. For $\ell = L-1$, the coefficient Ja^- in (6.2) determines the influence of the outflow boundary condition JU_L^n; using (6.5) and (6.7) there results:

$\underline{M > 1}$ $(\xi_x = 1)$: here, $\hat{Q}_x - J = 0$.

$\underline{M \leq 1}$ $(\xi_x = 0,-1)$: here, $\operatorname{rank}(\hat{Q}_x - J) = 1$.

For $\ell = 0$, the coefficient b^+ in (6.2) similarly determines the influence of the inflow boundary condition U_0^n. Now, using (6.7) and (6.8), there results

$\underline{M > 1}$ $(\xi_x = 1)$: $\quad \text{rank}(\hat{Q}_x + J) = 3,$

$\underline{M \leq 1}$ $(\xi_x = 0,-1)$: $\quad \text{rank}(\hat{Q}_x + J) = 2,$

i.e., rank $b^+ = 4$ $(M > 1)$, rank $b^+ = 3$ $(M \leq 1)$.

Thus, the number of boundary conditions for (3.9) which are effective when $\mu \to 0$ may be summarized as:

	Outflow	Inflow
$M > 1$	0	4
$M \leq 1$	1	3

These are exactly the number of boundary conditions which are appropriate for the Euler equations (6.1).

Summarizing, we have described a class of compact finite difference equations (3.9) for treating the Navier-Stokes equations when written in the form (2.12). For model problems in which the coefficient matrices appearing in these equations are symmetric and constant the resulting scheme can be shown to be convergent for all values of the mesh parameters $\lambda_x = \Delta t/\Delta x$, $\lambda_y = \Delta t/\Delta y$ and also to provide second-order accuracy. In this theory the influence of the viscosity μ primarily determines the size of the computational subdomains within which variations in the coefficient matrices A and B can be regarded as small.

An important feature of the finite difference scheme (3.9) is that the natural physical boundary conditions for the Navier-Stokes equations are employed; when $\mu \to 0$ only the boundary conditions for the Euler problem influence the solution.

Acknowledgments

An important motivation for the study described in Part I were questions raised in personal discussions with R. Beam and R. Warming concerning the related paper of Philips and Rose [4]. The study of the relationship of the scheme (2.12) to the Euler equations as described in section 6 of Part II was partially motivated by the work of MacCormack [3].

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